Optimizing Entanglement-Assisted Quantum Error Correcting Codes

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INTRODUCTION

- **Quantum Noise & the Stabilizer Solution**: Quantum information is sensitive to noise and decoherence, requiring robust error correction. Stabilizer codes encode logical qubits into larger physical systems using Pauli operators to detect and correct errors.
  - **A code [[n,k,d]] encodes k logical qubits into n physical ones, correcting up to t=⌊(d−1)/2⌋ errors.**
- **Limitations of Standard Approaches**: Traditional stabilizer codes face limitations/ mathematical conditions that restrict available code families and lead to inefficient resource usage for practical implementations.
- **Entanglement-Assisted Codes (EAQEC):** EAQEC codes [[n,k,d;c]] leverage c pre-shared Bell pairs (ebits) between sender and receiver to overcome limitations of standard stabilizer codes enabling:
  - *Flexibility in Code Construction*: Construction from any classical code.
  - *Implementation Feasibility*: Support for higher-rate communication and more flexible architectures for near-term quantum hardware.
  - *Resource Efficiency*: Entanglement reduces physical qubit requirements without sacrificing protection.
- ***Resource Reduction using EAQEC Example → Reducing 5 Qubits to 3 with 2 Ebits***: The canonical **[[5,1,3]] stabilizer code** (the smallest code that corrects arbitrary single-qubit errors) can be transformed into a **[[3,1,3;2]] entanglement-assisted code** using **only 3 physical qubits and 2 ebits.**

Code	n (physical qubits)	k (logical qubits)	d (distance)	c (ebits)	t Errors Corrected	Stabilizers	Notes
[[5,1,3]]	5	1	3	0	⌊(3-1)/2⌋=1	S = { XZZXI, IXZZX, XIXZZ, ZXIXZ}	Standard QECC
[[3,1,3;2]]	3	1	3	2	⌊(3-1)/2⌋=1	S = { XZZXI, ZZXIX, ZYYZI, YYZIZ}	EAQEC; 2 ebits allow for 2 fewer qubits

Code Equivalence via Entanglement Assistance

This transformation **(via row operations/Gaussian elimination** on the stabilizer matrix) **leverages entanglement** to reduce physical qubits. The [[3,1,3;2]] code, **despite fewer physical qubits (n)** and additional shared ebits (c), maintains error correction capability (t=1) by effectively replacing two ancilla qubits with the two shared ebits.

Key Insight: Correctable erasure sets allow *stabilizer support to be offloaded* onto shared entanglement, reducing qubit overhead.

MOTIVATION

- **The Entanglement Bottleneck**
  While powerful, EAQEC introduces new challenges because:
  - Generating high-quality entanglement requires specialized hardware.
  - Maintaining entangled states is experimentally demanding.
  - Current designs use fixed amounts of entanglement regardless of actual needs.
  - Resource costs scale poorly for large quantum systems.
- **Untapped Potential: Quantum Degeneracy**
  Many quantum codes exhibit degeneracy, where different error patterns produce identical syndromes. This property, often overlooked, enables:
  - Passive correction of certain error groups.
  - Reduced entanglement requirements through smarter resource allocation.
  - Simplified error correction circuits by eliminating redundant operations.

CONJECTURE

Core Idea: This conjecture formalizes a relationship between correctable erasures in stabilizer codes and the resource efficiency of entanglement-assisted (EA) codes. It shows that stabilizers *fully supported on a correctable set* reduce the entanglement cost of the resulting EA code, providing a *constructive method* to optimize EA codes.

Key Definitions:
- **Correctable Set (E):** A set of qubits recoverable after erasure.
- **Stabilizer Subgroup (S_E):** Subgroup of S where all stabilizers act *only* on *E*. If size $|S_E|= 2^s$, it has *s* independent generators.
- **Cleaning Lemma**: Ensuring correctable sets can be transformed for EA codes.

Let $\mathcal{C}$ be an $[\![n,k,d]\!]$ stabilizer code. Let E be a correctable set of qubits on $\mathcal{C}$, and let $\mathcal{S}_E$ be the subgroup of $\mathcal{S}$ with support on E, that is

$$\mathcal{S}_E = \{S \in \mathcal{S} \mid supp(S) \subseteq E\}.$$

Let $|E| = e$ and $|\mathcal{S}_E| = 2^s$. Then there exists an EA code with parameters $[\![n-e,k,d;e-s]\!]$.

That is, if there is a correctable set with a stabilizer supported on it we can reduce the number of entangled pairs we use. The goal is to prove this conjecture.

Implication: If a correctable set has stabilizers fully supported on it, the EA code requires **fewer entangled pairs** (*e−s* instead of *e*).

PROOF

The conjecture shows that for a correctable set E with S_E of size 2^s, we can:

ISOLATE
The s "redundant" stabilizers acting on E

CLEAN
Logical operators off E (via Cleaning Lemma)

REPURPOSE
Remaining e-s qubits into entanglement pairs

Critical Assumptions:
- ***Correctability of E***: Ensures logical information protection & enables Cleaning Lemma.
- ***S_E fully supported on E***: Ensures correct identification of s "redundant" stabilizers.

SETUP
- Determine a valid correctable set E and identify subgroup S_E.
- Apply Gaussian elimination on stabilizer generators to establish a basis for S_E.
- Apply Cleaning Lemma: Ensure logical info is preserved during manipulation.

VERIFY BY CONSTRUCTION
- Find a unitary transformation using local Clifford gates (CNOT/Hadamard gates).
- Remove redundant qubits and repurpose remaining qubits
- Verify and compare predicted EA code to final EA code ([[n,k,d]] → [[n-e,k,d;e-s]].

[[7,1,3]] STEANE CODE EXAMPLE

A pure [[7,1,3]] Steane code has distance 3 and corrects any 1-qubit error or 2-qubit erasure (i.e. detects 2 errors).

SETUP

Logical Operators: $\bar{X} = X^{\otimes 7}$ $\bar{Z} = Z^{\otimes 7}$

Correctable Set E: *Qubits* $\{4,5,6,7\}$ $(e=4)$

Subgroup S_E : Generated by $\langle S_1, S_2\rangle$ ($s=2$, since $|S_E| = 2^2$)

Stabilizer Generators:

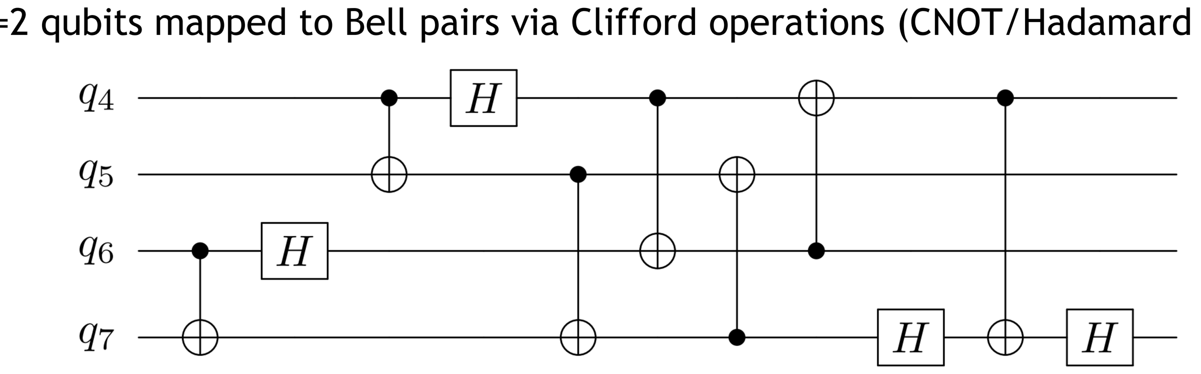
$$S_1 = I\otimes I\otimes I\otimes Z\otimes Z\otimes Z\otimes Z$$
$$S_2 = I\otimes I\otimes I\otimes X\otimes X\otimes X\otimes X$$
$$S_3 = X\otimes I\otimes X\otimes I\otimes X\otimes I\otimes X$$
$$S_4 = I\otimes X\otimes X\otimes I\otimes I\otimes X\otimes X$$
$$S_5 = Z\otimes I\otimes Z\otimes I\otimes Z\otimes I\otimes Z$$
$$S_6 = I\otimes Z\otimes Z\otimes I\otimes I\otimes Z\otimes Z$$

APPLY CONJECTURE

- **EA Code Parameters:** $[\![n-e,k,d;e-s]\!] \rightarrow$ **Predicted EA Code:** $[\![3,1,3;2]\!]$
- $|E| = 4, n = 7 \rightarrow$ Reduces to $n-e = 3$ physical qubits
- $|E| = 4, s = 2 \rightarrow$ EA code uses $e-s = 2$ Bell pairs

VERIFY BY CONSTRUCTION

Goal: Transform stabilizers to isolate E into s=2 stabilizers acting locally on E & e-s=2 qubits mapped to Bell pairs via Clifford operations (CNOT/Hadamard gates):



Resulting Stabilizers:

$$\left.\begin{matrix} S_1' = I\otimes I\otimes I\otimes I\otimes I\otimes Z\otimes I \\ S_2' = I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes Z \end{matrix}\right\} \text{Decoupled}$$

$$\left.\begin{matrix} S_3' = X\otimes I\otimes X\otimes I\otimes X\otimes I\otimes I \\ S_4' = I\otimes X\otimes X\otimes Z\otimes I\otimes I\otimes I \\ S_5' = Z\otimes I\otimes Z\otimes X\otimes I\otimes I\otimes I \\ S_6' = I\otimes Z\otimes Z\otimes I\otimes Z\otimes I\otimes I \end{matrix}\right\} \text{Half of Bell pairs shared}$$

Entangled Pairs:
e − s = 2 (qubits 4,5 now Bell pairs $\frac{1}{\sqrt{2}}$ (|00⟩+|11⟩) w/ Bob).

Final EA Code: [[3,1,3;2]] (preserves distance d=3 w/ 2 Bell pairs).

LIMITATIONS

Proof Completeness:
- Proof relies on heuristic Clifford operations (e.g., Steane code gates). A *general algorithm* for cleaning arbitrary stabilizers is needed.
- **Challenge:** Guaranteeing that e−s is always achievable for any correctable E.

Impure Code Caveats:
- Conjecture assumes well-defined S_E, but impure codes (e.g., Shor code) may have *partially* supported stabilizers on E.

Optimality Gaps:
- Optimality of *c=e−s* is unclear; there may exist codes where fewer pairs suffice.

FUTURE WORK

Generalizing the Proof (Further formalization)
- Develop a tool/*canonical form* for stabilizers in S_E to automate Clifford transformations.

Handling Edge Cases - Optimality Proof
- Extend conjecture to low-weight stabilizers, impure codes, etc.
- Prove *c ≥ e − s* for all codes, or identify further edge cases.

Experimental Validation
- Experimentally validate conjecture on small-scale quantum hardware (e.g., IBMQ) using:
  - **Metrics:** Entanglement consumption vs. recovery fidelity.
  - **Codes:** Steane, Shor, and surface codes, etc.